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LETTER TO THE EDITOR

$1/d$ expansions for critical amplitudes

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Abstract. We show how to expand *critical amplitudes* in inverse powers of the dimensionality d. The technique, which is rather general, is applied to the classical n-vector model, with particular emphasis on the self-avoiding walk ($n = 0$) and Ising ($n = 1$) cases. The existence of these expansions and of similar ones for critical points and critical constants suggests that $1/d$ expansions for *non-universal* quantities are analogous to the ε expansions for *universal* quantities.

Some years ago, Fisher and Gaunt (1964) showed how the *critical point* of the spin-$\frac{1}{2}$ Ising model with nearest-neighbour interactions and the self-avoiding walk (SAW) problem on a d-dimensional simple hypercubic lattice of coordination number $q = 2d$ may be expanded in inverse powers of d through $O(d^{-5})$. Their work was extended to the general classical n-vector model by Gerber and Fisher (1974), whose expansion contains the Ising ($n = 1$), SAW ($n = 0$) and spherical model ($n \to \infty$) results as special cases. Subsequently, analogous expansions have been derived for the critical points arising in a variety of lattice statistical problems including, for example, site and bond percolation processes, both undirected (Gaunt *et al* 1976, Gaunt and Ruskin 1978) and directed (Blease 1977), neighbour-avoiding walks (Gaunt *et al* 1984), trails (Guttmann 1985) and the growth parameters for site and bond trees (Gaunt *et al* 1982), c animals (Whittington *et al* 1983) and unrestricted animals (Gaunt *et al* 1976, Gaunt and Ruskin 1978).

It is also possible to derive $1/d$ expansions for some *critical constants*. For the Ising model, for example, one employs the expansion, given by Fisher and Gaunt (1964) (FG) through $O(d^{-5})$, for the reduced free energy, $\ln Z$, at some *arbitrary* temperature $T \geqslant T_c$. From this result one may readily obtain similar expansions for the reduced internal energy and entropy. Substituting the $1/d$ expansion for T_c yields $1/d$ expansions for the *critical* free energy, internal energy and entropy. Nath and Frank (1982) have given the resulting expansion for the reduced internal energy of the Ising (and spherical) model at its critical temperature through $O(d^{-5})$.

In this letter, we show, to our knowledge for the first time, how to expand *critical amplitudes* in inverse powers of d. Specifically, we consider the zero-field susceptibility of the classical n-vector model and derive an expansion for the critical amplitude correct through third order in $1/d$ and for general n. For the special cases $n = 1$ (Ising) and $n = 0$ (SAW), more complete information is available, and this enables us to calculate two additional terms, i.e. through $O(d^{-5})$.

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Consider, first, the Ising model ($n = 1$) for which the reduced zero-field susceptibility may be expanded as

$$\chi_0 = 1 + \sum_{l=1}^{\infty} a_l v^l$$

where $v = \tanh(J/kT)$ is a high temperature expansion variable. The coefficients a_l behave, asymptotically, like

$$a_l \approx A l^{\gamma-1} \omega^l \qquad (l \to \infty) \tag{1}$$

where $\omega = 1/v_c$, γ is the usual critical exponent and A is (apart from a constant factor) the corresponding critical amplitude of the susceptibility. For d greater than the critical dimension d_c (=4 for the Ising model), $\gamma = 1$ and the critical amplitude is given, more formally, by

$$A = \lim_{l\to\infty} (a_l/\omega^l) \qquad d > d_c. \tag{2}$$

FG have given expansions, valid for large d, of $a_l(d)$ and $\omega(d)$ in powers of $1/\sigma$, where $\sigma = 2d - 1$. Thus, from FG, equation (5.26), we have for $l \geq 11$,

$$a_l(d) = q\sigma^{l-1}[1 - (l-3)\sigma^{-2} - (3l-17)\sigma^{-3} + (\tfrac{1}{2}l^2 - 17\tfrac{1}{2}l + 128)\sigma^{-4} + (3l^2 - 108\tfrac{1}{3}l + 1072\tfrac{2}{3})\sigma^{-5} + \ldots] \tag{3}$$

and from FG, equation (5.28*b*),

$$\omega(d) = \sigma[1 - \sigma^{-2} - 3\sigma^{-3} - 14\sigma^{-4} - 79\tfrac{1}{3}\sigma^{-5} - \ldots]. \tag{4}$$

Remarkably, the ratio a_l/ω^l turns out to be independent of l and hence the limit of $l \to \infty$ in (2) follows trivially. We find

$$A_1 = (1 + \sigma^{-1})(1 + 3\sigma^{-2} + 17\sigma^{-3} + 128\sigma^{-4} + 1072\tfrac{2}{3}\sigma^{-5} + \ldots). \tag{5}$$

The subscript on A refers to the value of n.

The corresponding result for the SAW problem ($n = 0$) follows in a similar fashion. Expressions analogous to (3) and (4) are given by FG, equations (5.15) and (5.18*b*), respectively, and using these we obtain

$$A_0 = (1 + \sigma^{-1})(1 + 3\sigma^{-2} + 13\sigma^{-3} + 107\sigma^{-4} + 895\sigma^{-5} + \ldots). \tag{6}$$

The prefactor in (5) and (6) is the Bethe approximation for A, as may be seen from FG, equations (5.21) and (5.6), respectively.

For general n, we have used the results given by Gerber and Fisher (1974) (GF). In this case, $K(\equiv J/kT)$ is the appropriate high temperature expansion variable and for $d > d_c$ the critical amplitude is given by

$$A(n) = \lim_{l\to\infty} (a_l K_c^l). \tag{7}$$

In equation (3.15), GF have given a $1/q$ expansion for qK_c valid through fifth order and for general n. From this result we obtain the corresponding $1/\sigma$ expansion, namely,

$$\sigma K_c = 1 + \left(1 + \frac{n}{n+2}\right)\sigma^{-2} + \left(2 + \frac{3n}{n+2}\right)\sigma^{-3} + \left[12 + \frac{n}{n+2}\left(15 - \frac{12}{n+2} + \frac{8}{n+4}\right)\right]\sigma^{-4} + \left[66 + \frac{n}{n+2}\left(83 - \frac{76}{n+2} + \frac{40}{n+4} + \frac{12}{(n+2)^2}\right)\right]\sigma^{-5} + \ldots. \tag{8}$$

Using the reduced lattice constants and corresponding graphical weights given in (3.10)-(3.13) and table II of GF, we find

$$a_l/q\sigma^{l-1} = 1 - [(l-3) + (l-2)n/(n+2)]\sigma^{-2}$$
$$- [(2l-13) + 3(l-4)n/(n+2)]\sigma^{-3} - \dots \quad (9)$$

Unfortunately, we have been unable to write down the coefficients of σ^{-4} and σ^{-5} because, at these orders, GF do not give terms of order l^0 for some of the reduced lattice constants. It should be noted that such terms, although required for (9), do not contribute to the expression (8) for K_c.

By substituting (8) and (9) into (7) we obtain our most general result, namely, the critical amplitude of the classical n-vector model, given by

$$A(n) = (1+\sigma^{-1})\left[1 + \left(3 + \frac{2n}{n+2}\right)\sigma^{-2} + \left(13 + 12\frac{n}{n+2}\right)\sigma^{-3} + \dots\right] \quad (10)$$

correct through third order and for general n.

We note that for $n=0$, (10) reduces, as it must, to (6), i.e. $A(0) = A_0$. For $n=1$, (10) gives

$$A(1) = (1+\sigma^{-1})(1 + \tfrac{11}{3}\sigma^{-2} + 17\sigma^{-3} + \dots). \quad (11)$$

It is easy to show that the relation between the Ising amplitudes $A(1)$ and A_1 using the K and v variables, respectively, is (see, e.g., Essam and Hunter 1968)

$$A_1 = (\omega - \omega^{-1})K_c A(1). \quad (12)$$

Substituting (4), (11) and (8) (with $n=1$) into the right-hand side of (12) correctly reproduces (5) through $O(1/\sigma^3)$.

By combining (10) with a result from renormalisation group theory, we now derive a $1/\sigma$ expansion for the critical amplitude, $A'(n)$, of the *low temperature* susceptibility of the n-vector model. According to renormalisation group theory, the critical amplitude ratio $A(n)/A'(n)$ is a universal quantity. This implies that for $d \geq d_c$,

$$A(n)/A'(n) = R \quad (13)$$

where R, the ratio of the amplitudes in the mean-field approximation, is independent of σ. The $1/\sigma$ expansion for $A'(n)$ follows from (10) and (13) given that $R=2$ for the n-vector model. Thus,

$$A'(n) = \tfrac{1}{2}(1+\sigma^{-1})\left[1 + \left(3 + 2\frac{n}{n+2}\right)\sigma^{-2} + \left(13 + 12\frac{n}{n+2}\right)\sigma^{-3} + \dots\right]. \quad (14)$$

We now use the $1/\sigma$ expansions to obtain numerical estimates of critical amplitudes for $d > d_c$ and compare our results with high temperature series estimates obtained using standard series analysis techniques (Gaunt and Guttmann 1974). As pointed out by FG and GF, the $1/\sigma$ expansion for the critical point of the n-vector model is almost certainly asymptotic. The same is true of the $1/\sigma$ expansion for the critical amplitude. For the spherical model ($n \to \infty$), GF have shown rigorously that $1/\sigma$ expansions do not always yield optimum numerical values when truncated at the smallest term. Nevertheless, in the absence of anything better for $n=0$ and 1, we adopt either this criterion or truncation after the last term, and use an empirical procedure for estimating correction terms.

For the SAW problem, the terms in A_0 decrease monotonically when evaluated for $d=5$ and 6 and consequently we truncate the expansion after the last term in each case. For $d=5$, the last term is probably the smallest but this is less certain for $d=6$. Using this procedure we obtain

$$A_0^{(\sigma)} \simeq 1.207 \qquad (d=5) \tag{15a}$$

$$\simeq 1.143 \qquad (d=6) \tag{15b}$$

which estimates are only 5.0% and 1.3% smaller, respectively, than the best series estimates (Guttmann 1981)

$$A_0 = 1.27 \pm 0.02 \qquad (d=5) \tag{16a}$$

$$= 1.158 \pm 0.008 \qquad (d=6). \tag{16b}$$

The situation is precisely analogous for the Ising model and truncating the $1/\sigma$ expansions after the last term gives

$$A_1^{(\sigma)} \simeq 1.220 \qquad (d=5) \tag{17a}$$

$$\simeq 1.149 \qquad (d=6). \tag{17b}$$

The best series estimates (Guttmann 1981), namely,

$$A_1 = 1.311 \pm 0.009 \qquad (d=5) \tag{18a}$$

$$= 1.168 \pm 0.008 \qquad (d=6) \tag{18b}$$

are larger by only 6.9% and 1.6%, respectively.

Improved estimates of the amplitudes may be obtained from the $1/\sigma$ expansions by following an empirical procedure similar to one that proved successful for estimating critical points (Gaunt *et al* 1976, Gaunt and Ruskin 1978) and has worked satisfactorily in all the cases studied so far. Assume, as seems likely, that for $n=0$ and $d=5$ the last term in the $1/\sigma$ expansion is the smallest. To obtain an estimate identical with the series result (16*a*), we must add to the $1/\sigma$ expansion result f (=3.74) times the smallest term. Uncertainties in f of ± 1.19 will reproduce the uncertainties in (16*a*). Adopting the same procedure and the same value of f for $d=6$ and assuming the last term in the $1/\sigma$ expansion is again the smallest, we obtain

$$A_0^{(i)} = 1.165 \pm 0.007 \qquad (d=6). \tag{19}$$

This estimate, which is a considerable improvement over (15*b*), overlaps with the best series estimate (16*b*).

Adopting an analogous procedure for the Ising model, we find that for $d=5$ the best series estimate (18*a*) is reproduced using $f = 4.51 \pm 0.45$. This then leads to the improved estimate

$$A_1^{(i)} = 1.181 \pm 0.003 \qquad (d=6) \tag{20}$$

which only just fails to overlap with the best series estimate (18*b*).

In summary, we have shown how to expand critical amplitudes in inverse powers of the dimensionality. For the high temperature susceptibility amplitude of the classical n-vector model, we derive such an expansion correct through third order in $1/d$ and for general n. Two further terms are given for the special cases corresponding to the Ising ($n=1$) and SAW ($n=0$) problems. Good agreement has been obtained between numerical estimates of critical amplitudes obtained from the $1/d$ expansions and from the analysis of high temperature series. The low temperature susceptibility amplitude has a $1/d$ expansion which follows from the observation that the critical amplitude ratio is universal.

We mention that $1/d$ expansions may also be derived, using similar techniques, for the critical amplitudes associated with a number of other problems, e.g. the mean size of finite clusters in both site and bond percolation processes, the number of site and bond lattice animals, etc. Finally, we suggest that $1/d$ expansions for *non-universal* quantities (i.e. critical points, critical constants, critical amplitudes) should be viewed as analogous to the well known ε expansions which have been derived for *universal* quantities (i.e. critical exponents, critical amplitude ratios) using renormalisation group theory (e.g. Brézin *et al* 1976).

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